(b) Verify that if a uniformity $\mathcal{U}$ on a set X is induced by a metric ρ on the set X, then the uniformity $2^{\mathcal{U}}_{\mathcal{M}}$ on the family $\mathcal{M}$ of all bounded, non-empty closed subsets of (X,ρ) coincides with the uniformity induced by the Hausdorff metric.

(c) (Michael [1951]) Show that for every uniformity $\mathcal{U}$ on a topological space X, the topology on $\mathcal{Z}(X)$ induced by the uniformity $2^{\mathcal{U}}_{\mathcal{Z}(X)}$ coincides with the Vietoris topology.

(d) Verify that if the uniform space $(X,\mathcal{U})$ is totally bounded, then the space $(2^X, 2^{\mathcal{U}})$ also is totally bounded.

(e) Give an example of a complete uniform space $(X,\mathcal{U})$ such that the space $(2^X, 2^{\mathcal{U}})$ is not complete.

Hint. Consider the uniformity on the real line generated by the base consisting of all sets of the form $\bigcup\{A \times A : A \in \mathcal{A}\}$, where $\mathcal{A}$ is a countable cover of the real line by pairwise disjoint sets.

(f) Show that if the uniform space $(X,\mathcal{U})$ is compact, then the space $(2^X, 2^{\mathcal{U}})$ also is compact.

Cardinal functions IV (see Problems 1.7.12, 1.7.13, 2.7.9–2.7.11, 3.12.4, 3.12.7–3.12.11, 3.12.12(h) and 3.12.12(j))

8.5.17. The smallest cardinal number $\mathfrak{m} \geq \aleph_0$ such that on a Tychonoff space X there exists a uniformity of weight $\leq \mathfrak{m}$ is called the ***uniform weight*** of the space X and is denoted by $u(X)$.

(a) (Schaerf [1968]) Show that for every Tychonoff space X we have $w(X) = c(X)u(X)$.

Hint (Juhász [1971a]). Apply Remark 8.2.4.

(b) Observe that for every Tychonoff space X we have $w(X) = e(X)u(X)$ and deduce that $w(X) = f(X)u(X)$ for every cardinal function f in the diagram in Problem 3.12.7(e), except for $f = k$.

(c) Give an example of a topological space X of weight $\mathfrak{m}$ and of uniformities $\mathcal{U}_1$ and $\mathcal{U}_2$ on the space X such that $w(\mathcal{U}_1) > \mathfrak{m} > w(\mathcal{U}_2)$.

Strong inclusions

8.5.18 (Alexandroff and Ponomarev [1959]). Let X be a T_1-space and let $\mathcal{O}$ and $\mathcal{C}$ denote the families of all open and all closed subsets of X respectively. A relation $\ll$ between members of $\mathcal{C}$ and members of $\mathcal{O}$ is called a ***strong inclusion on the space*** X if it has properties (SI1)–(SI7) formulated in Section 8.4.

Show that for every T_1-space X there exists a one-to-one correspondence between proximities and strong inclusions on the space X.

Proximally continuous mappings of metric spaces

8.5.19 (Efremovič [1952]). (a) Let (X,ρ) and (Y,σ) be metric spaces and let δ and δ' denote the proximities induced by ρ and σ on X and Y respectively. Show that a mapping f of X to Y is proximally continuous with respect to δ and δ' if and only if f is uniformly continuous with respect to ρ and σ (cf. Exercise 8.1.A(a)).

Hint (Isbell [1964]). If a mapping f is not uniformly continuous, then there exist an $\epsilon > 0$ and two sequences $x_1, x_2, \ldots$ and $x'_1, x'_2, \ldots$ of points of X such that $\lim \rho(x_i, x'_i) = 0$ and $\sigma(f(x_i), f(x'_i)) \geq \epsilon$ for $i = 1, 2, \ldots$ Define an infinite set M of natural numbers such that $\sigma(f(x_i), f(x'_j)) \geq \epsilon/4$ for all $i, j \in M$.

(b) Note that two metric ρ_1 and ρ_2 on a set X are uniformly equivalent (see Exercise 4.1.B(b)) if and only if they induce the same proximity.

Uniformities and proximities

8.5.20 (Smirnov [1952]). Show that if a uniformity $\mathcal{U}$ on a space X induces a proximity δ on the space X, then the proximity $\mathcal{U}_0$ defined in Problem 8.5.7, the finest uniformity on the space X which is totally bounded and coarser than $\mathcal{U}$, also induces the proximity δ, i.e., the uniformity $\mathcal{U}_0$ coincides with the uniformity induced by the proximity δ induced by the uniformity $\mathcal{U}$. Observe that the Samuel compactification of a Tychonoff space X with respect to a uniformity $\mathcal{U}$ is the compactificaction which corresponds to the proximity δ induced by the uniformity $\mathcal{U}$ (cf. Theorem 8.4.13).

8.5.21 (Smirnov [1952]). Let δ be a proximity on a set X. A cover $\mathcal{A}$ of the set X is called *weakly δ-uniform* if there exists a sequence $\mathcal{A}_1, \mathcal{A}_2, \ldots$ of covers of X, where $\mathcal{A}_1 = \mathcal{A}$, satisfying the following two conditions:

(1) *$\mathcal{A}_{i+1}$ is a star refinement of $\mathcal{A}_i$ for $i = 1, 2, \ldots$*

(2) *For every pair A, B of subsets of X satisfying $A\delta B$ there exists for each i a set $A_i \in \mathcal{A}_i$ such that $A \cap A_i \neq \emptyset \neq B \cap A_i$.*

(a) Show that every δ-uniform cover of X is weakly δ-uniform.

(b) Prove that for every weakly δ-uniform cover $\mathcal{A}$ of X there exists a uniformity $\mathcal{U}$ on the set X which induces the proximity δ and contains the set $\bigcup\{A \times A : A \in \mathcal{A}\}$.

8.5.22 (Katětov [1959], Dowker [1961]). Verify that the collection $\mathbf{C}$ of all covers of the set $X = N \times N$, where N is the set of natural numbers, which have a refinement of the form $\{\{x\} \times A : x \in N, A \in \mathcal{A}\}$, where $\mathcal{A}$ is a finite cover of N by pairwise disjoint sets, has properties (UC1)–(UC4). Let $\mathcal{U}$ be the uniformity generated by $\mathbf{C}$ and let δ be the proximity on X induced by $\mathcal{U}$; check that $\Delta\delta[(N \times N) \setminus \Delta]$. Show that the covers $\{\{x\} \times N\}_{x \in N}$ and $\{N \times \{y\}\}_{y \in N}$ are both weakly δ-uniform and that the only common refinement of those covers is the cover consisting of all one-point subsets of X. Deduce that in the family of all uniformties on the set X which induce the proximity δ there is no finest uniformity.

8.5.23 (Smirnov [1952]). Prove that if the proximity δ on a set X is induced by a metric on X, then in the family of all uniformities on the set X which induce the proximity δ there exists a finest uniformity.

Hint. Show that if for an entourage V of the diagonal $\Delta \subset X \times X$ there exist two sequences $x_1, x_2, \ldots$ and $x'_1, x'_2, \ldots$ of points of X such that $|x_i - x'_i| \geq V$ for $i = 1, 2, \ldots$, then for any entourage W of the diagonal $\Delta \subset X \times X$ satisfying $4W \subset V$ there exists an infinite set M of natural numbers such that $|x_i - x'_j| \geq W$ for all $i, j \in M$.